

ON COHOMOLOGICALLY COMPLETE INTERSECTIONS

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ABSTRACT. An ideal I of a local Gorenstein ring (R, \mathfrak{m}) is called cohomologically complete intersection whenever $H_I^i(R) = 0$ for all $i \neq \text{height } I$. Here $H_I^i(R), i \in \mathbb{Z}$, denotes the local cohomology of R with respect to I . For instance, a set-theoretic complete intersection is a cohomologically complete intersection. Here we study cohomologically complete intersections from various homological points of view, in particular in terms of their Bass numbers of $H_I^c(R), c = \text{height } I$. As a main result it is shown that the vanishing $H_I^i(R) = 0$ for all $i \neq c$ is completely encoded in homological properties of $H_I^c(R)$, in particular in its Bass numbers.

INTRODUCTION

Let (R, \mathfrak{m}) denote a local Noetherian ring. For an ideal $I \subset R$ it is a rather difficult question to determine the smallest number $n \in \mathbb{N}$ of elements $a_1, \dots, a_n \in R$ such that $\text{Rad } I = \text{Rad}(a_1, \dots, a_n)R$. This number is called the arithmetic rank, $\text{ara } I$, of I . By Krull's generalized principal ideal theorem it follows that $\text{ara } I \geq \text{height } I$. Of a particular interest is the case whenever $\text{ara } I = \text{height } I$. In this situation I is called a set-theoretic complete intersection.

For the ideal I let $H_I^i(\cdot), i \in \mathbb{Z}$, denote the local cohomology functor with respect to I , see [2] for its definition and basic results. The cohomological dimension, $\text{cd } I$, defined by

$$\text{cd } I = \sup\{i \in \mathbb{Z} \mid H_I^i(R) \neq 0\}$$

is another invariant related to the ideal I . It is well known that $\text{height } I \leq \text{cd } I \leq \text{ara } I$. In particular, if I is set-theoretically a complete intersection it follows that $\text{height } I = \text{cd } I$, while the converse does not hold in general. Not so much is known about ideals with the property of $\text{height } I = \text{cd } I$. We call those ideals cohomologically complete intersections. In this paper we start with the investigations of cohomologically complete intersections, in particular when I is an ideal in a Gorenstein ring (R, \mathfrak{m}) .

As an application of our main results there is a characterization of cohomologically complete intersections for a certain class of ideals (cf. Theorem 0.1). A generalization to arbitrary ideals in a Gorenstein ring is shown in Section 3, namely: I is cohomologically a complete intersection if and only if a minimal injective resolution of $H_I^c(R), c = \text{height } I$, "looks like that of a Gorenstein ring" (see Theorem 3.2 for the precise statement).

Theorem 0.1. *Let I denote an ideal of a Gorenstein ring (R, \mathfrak{m}) with $c = \text{height } I$ and $d = \dim R/I$. Suppose that I is a complete intersection in $V(I) \setminus \{\mathfrak{m}\}$. Then the following conditions are equivalent:*

- (i) $H_I^i(R) = 0$ for all $i \neq c$, i.e. I is cohomologically a complete intersection.

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- (ii) $H_{\mathfrak{m}}^d(H_I^c(R)) \simeq E$ and $H_{\mathfrak{m}}^i(H_I^c(R)) = 0$, for all $i \neq d$, where E denotes the injective hull of the residue field R/\mathfrak{m} .
- (iii) The natural map $\text{Ext}_R^d(k, H_I^c(R)) \rightarrow k$ is isomorphic and $\text{Ext}_R^i(k, H_I^c(R)) = 0$ for all $i \neq d$.
- (iv) The Bass numbers of $H_I^c(R)$ satisfy

$$\dim_k \text{Ext}_R^i(k, H_I^c(R)) = \delta_{d,i}.$$

Moreover, if I satisfies the above conditions it follows that $\hat{R}^I \simeq \text{Hom}_R(H_I^c(R), H_I^c(R))$ and $\text{Ext}_R^i(H_I^c(R), H_I^c(R)) = 0$ for all $i \neq 0$, where \hat{R}^I denotes the I -adic completion of R .

It is a surprising fact – at least to the authors – that the information for the equality height $I = \text{cd } I$ is completely encoded in the cohomology module $H_I^c(R)$. Moreover the characterization of cohomologically complete intersections looks in a certain sense Gorenstein-like, cf. the well-known cohomological characterization of Gorenstein rings in terms of Bass numbers and Theorem 0.1 (iv). The “strong” characterization (cf. Theorem 0.1 (iv)) looks very similar to the “weak” characterization (cf. Theorem 0.1 (iii)). In fact it requires much more effort (it uses Matlis duals of local cohomology modules [6]). It is related to a Conjecture (cf. 2.7) that the map $\text{Ext}_R^d(k, H_I^c(R)) \rightarrow k$ is in general non-zero.

1. PRELIMINARIES

Let (R, \mathfrak{m}, k) denote a local Gorenstein ring with $n = \dim R$. In the following let $E = E_R(R/\mathfrak{m})$ denote the injective hull of the residue field $k = R/\mathfrak{m}$ as R -module.

A basic tool for the local cohomology with support in $\{\mathfrak{m}\}$ is the local duality theorem (cf. [2]). To this end let \hat{R} denote the completion of R and \hat{M} the completion of M .

Proposition 1.1. *For a finitely generated R -module M and an integer $i \in \mathbb{Z}$ there are the following natural isomorphisms:*

- (a) $H_{\mathfrak{m}}^i(M) \simeq \text{Hom}_R(\text{Ext}_R^{n-i}(M, R), E)$,
- (b) $\text{Ext}_{\hat{R}}^{n-i}(\hat{M}, \hat{R}) \simeq \text{Hom}_R(H_{\mathfrak{m}}^i(M), E)$.

In particular, $H_{\mathfrak{m}}^i(M)$ is an Artinian R -module as a consequence of Matlis duality. Moreover we have the following result.

Lemma 1.2. *Let I denote an ideal of height c in the Gorenstein ring (R, \mathfrak{m}) and $d = \dim R/I = n - c$. Then the following is true:*

- (a) *For all $i, j \in \mathbb{Z}$ there are isomorphisms*

$$\varprojlim \text{Ext}_R^i(\text{Ext}_R^j(R/I^\alpha, R), R) \simeq \text{Ext}_R^i(H_I^j(R), R),$$

where the inverse system on the left is induced by the natural projections.

- (b) *For all $i, j \in \mathbb{Z}$ there are natural isomorphisms*

$$\text{Ext}_{\hat{R}}^{n-i}(H_{I\hat{R}}^j(\hat{R}), \hat{R}) \simeq \text{Hom}_R(H_{\mathfrak{m}}^i(H_I^j(R)), E),$$

where \hat{R} denotes the completion of R .

- (c) $H_{\mathfrak{m}}^i(H_I^j(R)) = 0$ for all integers $i > n - j$ and $\text{Ext}_R^i(H_I^j(R), R) = 0$ for all $i < j$.

Proof. First of all let us recall that the Ext-functors in the first variable transform a direct system into an inverse system such that

$$\varprojlim \text{Ext}_R^i(M_\alpha, N) \simeq \text{Ext}_R^i(\varinjlim M_\alpha, N)$$

for any R -module N (cf. [13]). Now there is the isomorphism $H_I^j(R) \simeq \varinjlim \text{Ext}_R^j(R/I^\alpha, R)$ (cf. [2]). Therefore the first statement is shown to be true.

For the proof of the second statement first recall that

$$H_m^i(H_I^j(R)) \simeq \varinjlim H_m^i(\text{Ext}_R^j(R/I^\alpha, R))$$

since the local cohomology commutes with direct limits. Therefore the statement of (b) is a consequence of the local duality theorem and the previous observation on the inverse limit.

For the proof of the statement in (c) first recall that $c = \text{height } I = \text{height } I\hat{R}$. Therefore, because of Matlis duality it will be enough to prove that $\text{Ext}_R^i(H_I^j(R), R) = 0$ for all integers $i < j$. It is a well-known fact that for any $\alpha \in \mathbb{N}$ we have $\text{Ext}_R^j(R/I^\alpha, R) = 0$ for all $j < c$ and $j > n$. Moreover $\dim \text{Ext}_R^c(R/I^\alpha, R) = n - d$ and $\dim \text{Ext}_R^j(R/I^\alpha, R) \leq n - j$ for all $c < j \leq n$ (cf. [11, Proposition 2.3]).

Let $\mathfrak{b}_j = \text{Ann } \text{Ext}_R^j(R/I^\alpha, R)$. Therefore $\dim R/\mathfrak{b}_j \leq n - j$ and $\text{grade } \mathfrak{b}_j \geq j$ as easily seen. Therefore $\text{Ext}_R^i(N, R)$ vanishes for all R -modules N such that $\text{Supp}_R N \subseteq V(\mathfrak{b}_j)$ and all $i < j$ (cf. [2, Proposition 3.3]). Because of the statement in (a) this proves the vanishing of $\text{Ext}_R^i(H_I^j(R), R)$ for all $i < j$ and a given integer j . \square

As a technical tool in the next section we need a proposition on the behavior of the section functor on certain complexes of R -modules.

Proposition 1.3. *Let I be an ideal of a Noetherian ring R . Let X denote an arbitrary R -module with $\text{Supp}_R X \subset V(I)$. Then there is an isomorphism*

$$\text{Hom}_R(X, \Gamma_I(J)) \simeq \text{Hom}_R(X, J)$$

for any bounded complex J of injective R -modules.

Proof. Let M denote a finitely generated R -module. Then there is an isomorphism of complexes

$$\text{Hom}_R(M, \Gamma_I(J)) \simeq \Gamma_I(\text{Hom}_R(M, J))$$

(cf. [3]). Now let X be an arbitrary R -module with $\text{Supp}_R X \subset V(I)$. Then $X \simeq \varinjlim X_\alpha$ for a certain direct family of finitely generated submodules $X_\alpha \subset X$. Therefore $\text{Supp}_R X_\alpha \subset V(I)$ and

$$\text{Hom}_R(X, \Gamma_I(J)) \simeq \varprojlim \text{Hom}_R(X_\alpha, \Gamma_I(J)).$$

Because of the above result on finitely generated R -modules it implies

$$\text{Hom}_R(X_\alpha, \Gamma_I(J)) \simeq \text{Hom}_R(X_\alpha, J)$$

because any module in $\text{Hom}_R(X_\alpha, J)^i, i \in \mathbb{Z}$, has its support in $V(I)$. By passing to the limit this proves the claim. \square

As another technical tool we need a characterization of the vanishing of the Bass numbers of an arbitrary R -module X . Here (R, \mathfrak{m}) denotes a local ring with $k = R/\mathfrak{m}$ its residue field. Recall that the Bass numbers of X with respect to \mathfrak{m} are defined by

$$\dim_k \operatorname{Ext}_R^i(k, X), \text{ for all } i \in \mathbb{Z}.$$

Proposition 1.4. *Let (R, \mathfrak{m}) denote a local ring. Let X denote an arbitrary R -module. For an integer $s \in \mathbb{N}$ the following conditions are equivalent:*

- (i) $H_{\mathfrak{m}}^i(X) = 0$ for all $i < s$.
- (ii) $\operatorname{Ext}_R^i(k, X) = 0$ for all $i < s$.

If one of these equivalent conditions is satisfied there is an isomorphism

$$\operatorname{Ext}_R^s(k, X) \simeq \operatorname{Hom}_R(k, H_{\mathfrak{m}}^s(X)).$$

Proof. We proceed by an induction on s . If $s = 0$, then the equivalence is trivially true. Moreover, there is an isomorphism $\operatorname{Hom}_R(k, X) \simeq \operatorname{Hom}_R(k, H_{\mathfrak{m}}^0(X))$, as easily seen.

So let $s > 0$ and suppose that the claim is true for $s - 1$. Then there is an isomorphism

$$\operatorname{Ext}_R^{s-1}(k, X) \simeq \operatorname{Hom}_R(k, H_{\mathfrak{m}}^{s-1}(X)).$$

Because of $\operatorname{Supp}_R H_{\mathfrak{m}}^{s-1}(X) \subseteq V(\mathfrak{m})$ it follows that $\operatorname{Ext}_R^{s-1}(k, X) = 0$ if and only if $H_{\mathfrak{m}}^{s-1}(X) = 0$.

In order to complete the inductive step we have to prove the isomorphism of the statement. To this end consider the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_R^p(k, H_{\mathfrak{m}}^q(X)) \implies E_{\infty}^{p+q} = \operatorname{Ext}_R^{p+q}(k, X).$$

Because of $H_{\mathfrak{m}}^i(X) = 0$ for all $i < s$ and $\operatorname{Ext}_R^i(k, X) = 0$ for all $i < s$ it degenerates to the following isomorphisms

$$\operatorname{Hom}_R(k, H_{\mathfrak{m}}^s(X)) = E_2^{0,s} \simeq E_{\infty}^{0,s} \simeq E_{\infty}^s = \operatorname{Ext}_R^s(k, X).$$

This completes the proof. \square

In the core of the paper we are interested in cohomologically complete intersections I of a Gorenstein ring (R, \mathfrak{m}) . A technical necessary condition gives the following result.

Proposition 1.5. *Suppose that I denotes an ideal of a Gorenstein (R, \mathfrak{m}) such that $H_I^i(R) = 0$ for all $i \neq c$, $c = \operatorname{height} I$. Then R/I is unmixed, i.e. $c = \operatorname{height} IR_{\mathfrak{p}}$ for all minimal prime ideals $\mathfrak{p} \in V(I)$.*

Proof. Suppose there exists a prime ideal \mathfrak{p} minimal in $V(I)$ such that $h = \operatorname{height} IR_{\mathfrak{p}} > c$. Then $IR_{\mathfrak{p}}$ is $\mathfrak{p}R_{\mathfrak{p}}$ -primary and therefore

$$H_I^h(R) \otimes_R R_{\mathfrak{p}} \simeq H_{IR_{\mathfrak{p}}}^h(R_{\mathfrak{p}}) \neq 0.$$

But this means that $H_I^h(R) \neq 0$, $h > c$, in contradiction to the assumption. \square

As a further technical tool we shall prove a result for complexes which is well-known in the case of a module.

Proposition 1.6. *Let R denote a Noetherian ring and $I \subset R$ an ideal. Let X^\cdot denote a complex of R -modules such that $\operatorname{Supp} H^i(X^\cdot) \subseteq V(I)$ for all $i \in \mathbb{Z}$. Then $H_I^i(X^\cdot) \simeq H^i(X^\cdot)$ for all $i \in \mathbb{Z}$.*

Proof. In order to compute the hypercohomology of the complex $R\Gamma_I(X^\cdot)$ there is the following spectral sequence

$$E_2^{p,q} = H_I^p(H^q(X^\cdot)) \implies E_\infty^{p+q} = H_I^{p+q}(X^\cdot).$$

By the assumption $\text{Supp } H^i(X^\cdot) \subseteq V(I)$ so that $E_2^{p,q} = 0$ for all $p \neq 0$. This implies a partial degeneration of the spectral sequence to the isomorphisms

$$H_I^i(X^\cdot) \simeq H_I^0(H^i(X^\cdot)) \simeq H^i(X^\cdot)$$

for all $i \in \mathbb{Z}$. □

As another auxiliary tool in the following we need the so-called Hartshorne-Lichtenbaum vanishing theorem. We give it here for a slight sharpened form as we shall need it. A proof can e. g. be found in [6, 8.1.9 and 8.2.1].

Proposition 1.7. *Let (R, \mathfrak{m}) denote a local Gorenstein ring with $n = \dim R$. Let I be an ideal of R . Let M be a finitely generated R -module. Then*

$$\text{Ass}_{\hat{R}} \text{Hom}_R(H_I^n(M), E) = \{\mathfrak{p} \in \text{Ass}_{\hat{R}} \hat{M} \mid \dim \hat{R}/\mathfrak{p} = n \text{ and } \dim \hat{R}/(\mathfrak{p} + I\hat{R}) = 0\}.$$

As an epimorphic image of $H_{\mathfrak{m}}^n(M)$, $H_I^n(M)$ is an Artinian R -module. Here \hat{M} denotes the completion of M .

2. THE TRUNCATION COMPLEX

In this section let (R, \mathfrak{m}) denote a local Gorenstein ring and $d = \dim R$. Let $R \xrightarrow{\sim} E^\cdot$ denote a minimal injective resolution of R as an R -module. It is a well-known fact that

$$E^i \simeq \bigoplus_{\mathfrak{p} \in \text{Spec } R, \text{height } \mathfrak{p}=i} E_R(R/\mathfrak{p}),$$

where $E_R(R/\mathfrak{p})$ denotes the injective hull of R/\mathfrak{p} . See [1] for these and related results on Gorenstein rings.

Now let $I \subset R$ denote an ideal and $c = \text{height } I$. Then $d = \dim R/I = n - c$. The local cohomology modules $H_I^i(R), i \in \mathbb{Z}$, are – by definition – the cohomology modules of the complex $\Gamma_I(E^\cdot)$. Because of $\Gamma_I(E_R(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin V(I)$ it follows that $\Gamma_I(E^\cdot)^i = 0$ for all $i < c$.

Therefore $H_I^c(R) = \text{Ker}(\Gamma_I(E^\cdot)^c \rightarrow \Gamma_I(E^\cdot)^{c+1})$. This observation provides an embedding $H_I^c(R)[-c] \rightarrow \Gamma_I(E^\cdot)$ of complexes of R -modules.

Definition 2.1. The cokernel of the embedding $H_I^c(R)[-c] \rightarrow \Gamma_I(E^\cdot)$ is defined as $C_R(I)$, the truncation complex. So there is a short exact sequence of complexes of R -modules

$$0 \rightarrow H_I^c(R)[-c] \rightarrow \Gamma_I(E^\cdot) \rightarrow C_R(I) \rightarrow 0.$$

In particular it follows that $H^i(C_R(I)) = 0$ for $i \leq c$ or $i > d$ and $H^i(C_R(I)) \simeq H_I^i(R)$ for $c < i \leq d$.

The advantage of the truncation complex is that it separates information of the local cohomology modules $H_I^i(R), i = c$, from those with $i \neq c$. A first result in this direction is the following lemma.

Lemma 2.2. *With the previous notation there are an exact sequence*

$$0 \rightarrow H_{\mathfrak{m}}^{n-1}(C_R(I)) \rightarrow H_{\mathfrak{m}}^d(H_I^c(R)) \rightarrow E \rightarrow H_{\mathfrak{m}}^n(C_R(I)) \rightarrow 0,$$

isomorphisms $H_{\mathfrak{m}}^{i-c}(H_I^c(R)) \simeq H_{\mathfrak{m}}^{i-1}(C_R(I))$ for $i < n$ and the vanishing $H_{\mathfrak{m}}^{i-c}(H_I^c(R)) = 0$ for $i > n$.

Proof. Take the short exact sequence of the truncation complex (cf. 2.1) and apply the derived functor $R\Gamma_{\mathfrak{m}}(\cdot)$. In the derived category this provides a short exact sequence of complexes

$$0 \rightarrow R\Gamma_{\mathfrak{m}}(H_I^c(R))[-c] \rightarrow R\Gamma_{\mathfrak{m}}(\Gamma_I(E)) \rightarrow R\Gamma_{\mathfrak{m}}(C_R(I)) \rightarrow 0.$$

Since $\Gamma_I(E)$ is a complex of injective R -modules we might use $\Gamma_{\mathfrak{m}}(\Gamma_I(E))$ as a representative of $R\Gamma_{\mathfrak{m}}(\Gamma_I(E))$. But now there is an equality for the composite of section functors $\Gamma_{\mathfrak{m}}(\Gamma_I(\cdot)) = \Gamma_{\mathfrak{m}}(\cdot)$. Therefore $\Gamma_{\mathfrak{m}}(E)$ is a representative of $R\Gamma_{\mathfrak{m}}(\Gamma_I(E))$ in the derived category. But now $\Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{p})) = 0$ for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$ while $\Gamma_{\mathfrak{m}}(E) \simeq E$. So there is an isomorphism of complexes $\Gamma_{\mathfrak{m}}(E) \simeq E[-n]$.

With these observations in mind the above short exact sequence induces the exact sequence of the statement and the isomorphisms

$$H_{\mathfrak{m}}^{i-c}(H_I^c(R)) \simeq H_{\mathfrak{m}}^{i-1}(C_R(I)), \quad i < n$$

by view of the corresponding long exact cohomology sequence. Moreover the vanishing of $H_{\mathfrak{m}}^i(H_I^c(R))$ for all $i > d$ is shown above (cf. 1.2). \square

As a consequence there is the following necessary condition for an ideal $I \subset R$ to be a cohomologically complete intersection. As we shall see later this is not sufficient (cf. 4.1).

Corollary 2.3. *Let $I \subset R$ be an ideal with height $I = c$. Suppose that $H_I^i(R) = 0$ for all $i \neq c$. Then $H_{\mathfrak{m}}^d(H_I^c(R)) \simeq E$ and $H_{\mathfrak{m}}^i(H_I^c(R)) = 0$ for all $i \neq c$.*

Proof. By the assumption we have the vanishing of $H_I^i(R)$ for all $i \neq c$. Therefore the truncation complex $C_R(I)$ is bounded and homologically trivial. In order to compute the hypercohomology $H_{\mathfrak{m}}^i(C_R(I))$ consider the following spectral sequence

$$E_2^{p,q} = H_{\mathfrak{m}}^p(H^q(C_R(I))) \implies E_{\infty}^{p+q} = H_{\mathfrak{m}}^{p+q}(C_R(I)).$$

Because all the initial terms vanish and because of the finiteness of the spectral sequence $H_{\mathfrak{m}}^i(C_R(I)) = 0$ for all $i \in \mathbb{Z}$. So the claim is true by Lemma 2.2. \square

Let us continue with a result – in a certain sense – dual to the statement of Lemma 2.2. Here let \hat{R}^I denote the I -adic completion of R .

Lemma 2.4. *With the above notation the following is true:*

(a) *There is a short exact sequence*

$$0 \rightarrow \text{Ext}_R^0(C_R(I), R) \rightarrow \hat{R}^I \rightarrow \text{Ext}_R^c(H_I^c(R), R) \rightarrow \text{Ext}_R^1(C_R(I), R) \rightarrow 0.$$

(b) *There are isomorphisms $\text{Ext}_R^{i+c}(H_I^c(R), R) \simeq \text{Ext}_R^{i+1}(C_R(I), R)$ for $i > 0$.*

(c) *We have the vanishing $\text{Ext}_R^i(C_R(I), R) = 0$ for $i < 0$.*

Proof. Let $R \xrightarrow{\sim} E^\cdot$ denote the minimal injective resolution of the Gorenstein ring R . Apply the functor $\text{Hom}(\cdot, E^\cdot)$ to the short exact sequence of the truncation complex as defined in 2.1. Since E^\cdot is a complex of injective R -modules it provides a short exact sequence of complexes

$$0 \rightarrow \text{Hom}_R(C_R(I), E^\cdot) \rightarrow \text{Hom}_R(\Gamma_I(E^\cdot), E^\cdot) \rightarrow \text{Hom}(H_I^c(R), E^\cdot)[c] \rightarrow 0.$$

By the definition the cohomology modules of the complexes on the left resp. on the right are $\text{Ext}_R^i(C_R(I), R)$ resp. $\text{Ext}_R^{i-c}(H_I^c(R), R)$ for all $i \in \mathbb{Z}$.

Let us consider the complex in the middle. Let $\underline{y} = y_1, \dots, y_s$ be a generating set of the ideal I . Then $\Gamma_I(E^\cdot) \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes E^\cdot$, where $\check{C}_{\underline{y}}$ denotes the Čech complex with respect to \underline{y} (cf [10, Theorem 1.1] for the details).

Therefore $\text{Hom}_R(\Gamma_I(E^\cdot), E^\cdot) \simeq \text{Hom}_R(\check{C}_{\underline{y}}, \text{Hom}_R(E^\cdot, E^\cdot))$. Because of the quasi-isomorphism $R \xrightarrow{\sim} \text{Hom}_R(E^\cdot, E^\cdot)$, recall that E^\cdot is a dualizing complex of R . Therefore the last complex is a representative of $\text{R Hom}_R(\check{C}_{\underline{y}}, R)$ in the derived category. But it follows that

$$\text{R Hom}_R(\check{C}_{\underline{y}}, R) \xrightarrow{\sim} \hat{R}^I,$$

as shown in [10, Theorem 1.1]. Therefore with the above considerations the long exact cohomology sequence provides the statements (a) and (b) of the claim. With the aid of 1.2 this proves also the statement in (c). \square

As an application there is another necessary criterion for an ideal $I \subset R$ to be cohomologically a complete intersection.

Corollary 2.5. *Let $I \subset R$ denote an ideal of height $I = c$. Suppose that $H_I^i(R) = 0$ for all $i \neq c$. Then there is an isomorphism $\hat{R}^I \simeq \text{Ext}_R^c(H_I^c(R), R)$ and the vanishing $\text{Ext}_R^{i+c}(H_I^c(R), R) = 0$ for all $i \neq 0$.*

Proof. By virtue of the Lemma 2.4 It will be enough to prove that $\text{Ext}_R^i(C_R(I), R)$ vanishes for all $i \neq 0$. Let $R \xrightarrow{\sim} E^\cdot$ denote the minimal injective resolution of R . Then $\text{Hom}_R(C_R(I), E^\cdot)$ is a representative of $\text{Ext}_R^i(C_R(I), R)$ in the derived category. By the assumption $C_R(I)$ is an exact complex. So the conclusion follows because $\text{Hom}_R(\cdot, E^\cdot)$ preserves exactness. \square

In the next result there is a consideration of the Bass numbers of the local cohomology modules $H_I^c(R)$. This provides a certain necessary numerical condition for I to be a cohomologically complete intersection as we shall see later.

Lemma 2.6. *With the above notation there is a natural homomorphism*

$$\phi : \text{Ext}_R^d(k, H_I^c(R)) \rightarrow k.$$

In addition, suppose that $H_I^i(R) = 0$ for all $i \neq c$. Then ϕ is an isomorphism and $\text{Ext}_R^i(k, H_I^c(R)) = 0$ for all $i \neq d$.

Proof. Apply the derived functor $\text{R Hom}_R(k, \cdot)$ to the short exact sequence as it is defined in the definition of the truncation complex (cf. 2.1). Then there is the following short exact sequence of complexes in the derived category

$$0 \rightarrow \text{R Hom}_R(k, H_I^c(R))[-c] \rightarrow \text{R Hom}_R(k, \Gamma_{\mathfrak{m}}(E^\cdot)) \rightarrow \text{R Hom}_R(k, C_R(I)) \rightarrow 0.$$

Now we consider the complex in the middle. It is represented by $\mathrm{Hom}_R(k, \Gamma_{\mathfrak{m}}(E^\cdot))$ since $\Gamma_{\mathfrak{m}}(E^\cdot)$ is a complex of injective modules. Moreover there are the following isomorphisms

$$\mathrm{Hom}_R(k, \Gamma_{\mathfrak{m}}(E^\cdot)) \simeq \mathrm{Hom}_R(k, E^\cdot) \simeq E[-n].$$

By virtue of the long exact cohomology sequence it yields the natural homomorphism of the statement. Under the additional assumption of $H_I^i(R) = 0$ for all $i \neq c$ it follows that $C_R(I)$ is an exact complex. Therefore $\mathrm{RHom}_R(k, C_R(I))$ is also an exact complex. Whence the long exact cohomology sequence applied to the above short exact sequence provides the statements on the Bass numbers of $H_I^c(R)$. \square

Conjecture 2.7. It is an open problem whether the natural homomorphism

$$\phi : \mathrm{Ext}_R^d(k, H_I^c(R)) \rightarrow k$$

is in general non-zero, that is a surjection. In general the k -vector space $\mathrm{Ext}_R^d(k, H_I^c(R))$ is not of finite dimension (cf. Example 4.2).

In the next we are interested in the endomorphism ring of $H_I^c(R)$. As a consequence it provides another necessary condition for an ideal I to be a cohomologically complete intersection.

Lemma 2.8. *Let $I \subset R$ denote an ideal of a Gorenstein ring (R, \mathfrak{m}) and $c = \mathrm{height} I$. Then there is a natural isomorphism*

$$\mathrm{Hom}_R(H_I^c(R), H_I^c(R)) \simeq \mathrm{Ext}_R^c(H_I^c(R), R).$$

Moreover, suppose that $H_I^i(R) = 0$ for all $i \neq c$. Then $\hat{R}^I \simeq \mathrm{Hom}_R(H_I^c(R), H_I^c(R))$ and $\mathrm{Ext}_R^i(H_I^c(R), H_I^c(R)) = 0$ for all $i \neq 0$.

Proof. Consider the short exact sequence of complexes as introduced by the definition of the truncation complex in 2.1. Apply the derived functor $\mathrm{RHom}_R(H_I^c(R), \cdot)$ to this sequence. So, in the derived category there is a short exact sequence of complexes

$$\begin{aligned} 0 \rightarrow \mathrm{RHom}_R(H_I^c(R), H_I^c(R))[-c] \rightarrow \mathrm{RHom}_R(H_I^c(R), \Gamma_I(E^\cdot)) \rightarrow \\ \rightarrow \mathrm{RHom}_R(H_I^c(R), C_R(I)) \rightarrow 0. \end{aligned}$$

Because $\Gamma_I(E^\cdot)$ is a complex of injective R -modules a representative for the complex in the middle is given by $\mathrm{Hom}_R(H_I^c(R), \Gamma_I(E^\cdot))$. Now $\mathrm{Supp}_R(H_I^c(R)) \subset V(I)$. Whence Proposition 1.3 shows that this complex is isomorphic to $\mathrm{Hom}_R(H_I^c(R), E^\cdot)$. So the above short exact sequence of complexes induces an exact sequence

$$\begin{aligned} \mathrm{Ext}_R^{c-1}(H_I^c(R), C_R(I)) \rightarrow \mathrm{Hom}_R(H_I^c(R), H_I^c(R)) \rightarrow \\ \rightarrow \mathrm{Ext}_R^c(H_I^c(R), R) \rightarrow \mathrm{Ext}_R^c(H_I^c(R), C_R(I)). \end{aligned}$$

In order to finish the proof of the first statement it will be enough to show that $\mathrm{Ext}_R^i(H_I^c(R), C_R(I)) = 0$ for $i = c - 1, c$.

To this end let $C_R(I) \xrightarrow{\sim} F^\cdot$ denote an injective resolution of the complex $C_R(I)$ (cf. [3]). Then by definition it follows that $H^i(F^\cdot) = 0$ for all $i \leq c$ and all $i > n$ resp. $H^i(F^\cdot) \simeq H_I^i(R)$ for $c < i \leq n$. Moreover,

$$\mathrm{Ext}_R^i(H_I^c(R), C_R(I)) \simeq H^i(\mathrm{Hom}_R(H_I^c(R), F^\cdot)).$$

In order to compute the cohomology of the complex $\text{Hom}_R(H_I^c(R), F^\cdot)$ there is the following spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(H_I^c(R), H^q(F^\cdot)) \implies E_\infty^{p+q} = H^{p+q}(\text{Hom}_R(H_I^c(R), F^\cdot)).$$

Let $p + q \leq c$. For $p \geq 0$ it follows that $q \leq c$. Therefore it turns out that

$$\text{Ext}_R^p(H_I^c(R), H^q(F^\cdot)) = 0 \text{ for all } p + q \leq c.$$

So that $H^i(\text{Hom}_R(H_I^c(R), F^\cdot)) = 0$ for all $i \leq c$ as a consequence of the spectral sequence. This proves the first isomorphism of the statement.

Now assume that $H_I^i(R) = 0$ for all $i \neq c$. Then the complex $\text{Hom}_R(H_I^c(R), F^\cdot)$ is exact. So there are isomorphisms

$$\text{Ext}_R^{i-c}(H_I^c(R), H_I^c(R)) \simeq \text{Ext}_R^i(H_I^c(R), R) \text{ for all } i \in \mathbb{Z}.$$

By view of Corollary 2.5 this completes the proof. \square

The previous result is a slight extension of results of the first author and Stückrad (cf. [8, 2.2 (iii)]). There it is shown that the endomorphism ring of $H_I^c(R)$ is isomorphic to R for a cohomologically complete intersection ideal I in a complete local Gorenstein ring (R, \mathfrak{m}) .

Moreover, the previous result has an interesting application. It implies the non-vanishing of a certain local cohomology module of $H_I^c(R)$, $c = \text{height } I$.

Corollary 2.9. *Let $I \subset R$ denote an ideal of a Gorenstein ring (R, \mathfrak{m}) and $c = \text{height } I$. Then $H_{\mathfrak{m}}^d(H_I^c(R)) \neq 0$ where $d = \dim R/I$.*

Proof. By view of Lemma 1.2 (b) it will be enough to show that $\text{Ext}_{\hat{R}}^c(H_{I\hat{R}}^c(\hat{R}), \hat{R})$ does not vanish. Let us assume that R is a complete local ring. By virtue of Corollary 2.8 there is the following isomorphism $\text{Hom}_R(H_I^c(R), H_I^c(R)) \simeq \text{Ext}_R^c(H_I^c(R), R)$. Because of $H_I^c(R) \neq 0$ the endomorphism ring of $H_I^c(R)$ is non-trivial. \square

3. MAIN RESULTS

In this section let (R, \mathfrak{m}) denote a n -dimensional Gorenstein ring. Let $I \subset R$ be an ideal with $c = \text{height } I$ and $\dim R/I = n - c$. Then we shall prove our first characterization of cohomologically complete intersections. To this end let us fix the abbreviation $h(\mathfrak{p}) = \dim R_{\mathfrak{p}} - c$ for a prime ideal $\mathfrak{p} \in V(I)$.

Theorem 3.1. *With the previous notation the following conditions are equivalent:*

- (i) $H_I^i(R) = 0$ for all $i \neq c$, i. e. I is a cohomologically complete intersection.
- (ii) For all $\mathfrak{p} \in V(I)$ the natural map

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^{h(\mathfrak{p})}(H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) \rightarrow E(k(\mathfrak{p}))$$

is an isomorphism and $H_{\mathfrak{p}R_{\mathfrak{p}}}^i(H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) = 0$ for all $i \neq h(\mathfrak{p})$.

- (iii) For all $\mathfrak{p} \in V(I)$ the natural map

$$H_{\widehat{\mathfrak{p}R_{\mathfrak{p}}}}^{h(\mathfrak{p})}(H_{\widehat{IR_{\mathfrak{p}}}}^c(\widehat{R_{\mathfrak{p}}})) \rightarrow E(k(\mathfrak{p}))$$

is an isomorphism and $H_{\widehat{\mathfrak{p}R_{\mathfrak{p}}}}^i(H_{\widehat{IR_{\mathfrak{p}}}}^c(\widehat{R_{\mathfrak{p}}})) = 0$ for all $i \neq h(\mathfrak{p})$.

(iv) For all $\mathfrak{p} \in V(I)$ the natural map

$$\widehat{R}_{\mathfrak{p}} \rightarrow \text{Ext}_{\widehat{R}_{\mathfrak{p}}}^c(H_{I\widehat{R}_{\mathfrak{p}}}^c(\widehat{R}_{\mathfrak{p}}), \widehat{R}_{\mathfrak{p}})$$

is an isomorphism and $\text{Ext}_{\widehat{R}_{\mathfrak{p}}}^i(H_{I\widehat{R}_{\mathfrak{p}}}^c(\widehat{R}_{\mathfrak{p}}), \widehat{R}_{\mathfrak{p}}) = 0$ or all $i \neq c$.

(v) For all $\mathfrak{p} \in V(I)$ the natural map

$$\text{Ext}_{R_{\mathfrak{p}}}^{h(\mathfrak{p})}(k(\mathfrak{p}), H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) \rightarrow k(\mathfrak{p})$$

is an isomorphism and $\text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) = 0$ for all $i \neq h(\mathfrak{p})$.

Proof. (i) \Rightarrow (ii): Let $\mathfrak{p} \in V(I)$. Then $\text{height } IR_{\mathfrak{p}} = c$ (cf. Proposition 1.5) and $h(\mathfrak{p}) = \dim R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ as easily seen. Clearly $H_{IR_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) = 0$ for all $i \neq c = \text{height } IR_{\mathfrak{p}}$. Therefore the statement turns out by virtue of Corollary 2.3.

(ii) \Leftrightarrow (iii): This equivalence is a consequence of the faithful flatness of $R_{\mathfrak{p}} \rightarrow \widehat{R}_{\mathfrak{p}}$ and the fact that local cohomology commutes with flat extensions.

(iii) \Leftrightarrow (iv): By Matlis duality this equivalence follows by Lemma 1.2 (b). Recall that $h(\mathfrak{p}) = \dim R_{\mathfrak{p}} - c$ by definition.

(ii) \Leftrightarrow (v): Since both of the conditions localize it will be enough to prove the equivalence for the maximal ideal of a local Gorenstein ring (R, \mathfrak{m}) . Then the equivalence of the vanishings follow (cf. Proposition 1.4) for $X = H_I^c(R)$ and all $i < d$. Moreover it provides that the natural map $\text{Ext}_R^d(k, H_I^c(R)) \rightarrow \text{Hom}_R(k, H_{\mathfrak{m}}^d(H_I^c(R)))$ is an isomorphism. There is a commutative diagram

$$\begin{array}{ccc} \text{Ext}_R^d(k, H_I^c(R)) & \rightarrow & k \\ \downarrow & & \downarrow \\ \text{Hom}_R(k, H_{\mathfrak{m}}^d(H_I^c(R))) & \rightarrow & k. \end{array}$$

By the construction (cf. Lemma 2.2 and Lemma 2.6) it turns out that the second vertical map is the identity. Therefore $\text{Ext}_R^d(k, H_I^c(R)) \rightarrow k$ is an isomorphism if and only if $\text{Hom}_R(k, H_{\mathfrak{m}}^d(H_I^c(R))) \rightarrow k$ is an isomorphism. Now suppose that $H_{\mathfrak{m}}^d(H_I^c(R)) \rightarrow E$ is an isomorphism. Then it follows easily that $\text{Ext}_R^d(k, H_I^c(R)) \rightarrow k$ is an isomorphism. The converse follows by Theorem 3.2 in a more general context.

(ii) \Rightarrow (i): We proceed by induction on $d = \dim R/I$. In the case of $d = 0$ the ideal I is \mathfrak{m} -primary. Therefore the statement is true because R is a Gorenstein ring. So let $d > 0$. By view of Corollary 2.9 $\dim R_{\mathfrak{p}} - c = \dim R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ and therefore $c = \text{height } IR_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$. By induction hypothesis it follows that $H_{IR_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) = 0$ for all $i \neq c$ and all $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$. This means that $\text{Supp } H_I^i(R) \subset \{\mathfrak{m}\}$ for all $i \neq c$. Therefore

$$H_{\mathfrak{m}}^i(C_R(I)) \simeq H^i(C_R(I)) \simeq H_I^i(R) \text{ for } c < i \leq n$$

and $H_{\mathfrak{m}}^i(C_R(I)) = 0$ for $i \leq c$ and $i > n$. Because of the assumption for $\mathfrak{p} = \mathfrak{m}$, the maximal ideal, it follows that $H_I^i(R) = 0$ for all $i \neq c$ (cf. Lemma 2.2). \square

Before we shall go into the details of the proof of Theorem 0.1 we have to complete the proof of previous Theorem 3.1 in the light of the Conjecture 2.7. To be more precise we shall prove the following important result.

Theorem 3.2. *With the notion from the beginning of this section the following conditions are equivalent:*

- (i) $H_I^i(R) = 0$ for all $i \neq c$, i. e. I is a cohomologically complete intersection.
- (ii) For every $\mathfrak{p} \in V(I)$ it holds

$$\dim_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), H_{I_{R_{\mathfrak{p}}}}^c(R_{\mathfrak{p}})) = \delta_{h(\mathfrak{p}), i}$$

for all $i \in \mathbb{Z}$.

Proof. The implication (i) \Rightarrow (ii) is a consequence of the previous Theorem 3.1. In order to prove the reverse implication (ii) \Rightarrow (i) we proceed by induction on $d = \dim R/I$. Because R is a Gorenstein ring the claim is obviously true for $d = 0$. For the next let $d = 1$, i.e. $n = c + 1$. Then $\text{Supp}_R H_I^n(R) \subset \{\mathfrak{m}\}$ and we have to show $H_I^n(R) = 0$. There is the following spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(R/\mathfrak{m}, H_I^q(R)) \Rightarrow E_{\infty}^{p+q} = \text{Ext}_R^{p+q}(R/\mathfrak{m}, R)$$

and, as a part of it, the boundary homomorphism

$$E_2^{0,n} = \text{Hom}_R(R/\mathfrak{m}, H_I^n(R)) \xrightarrow{d} E_2^{2,c} = \text{Ext}_R^2(R/\mathfrak{m}, H_I^c(R)) = 0.$$

Moreover there is the isomorphism $\text{Ext}_R^1(R/\mathfrak{m}, H_I^c(R)) = E_2^{1,c} \simeq E_{\infty}^{1,c}$. Recall that all rows except those with $q = c, c + 1$ are zero. By the hypothesis this is a one-dimensional k -vector space. Moreover $E_{\infty}^n = \text{Ext}_R^n(k, R)$ is also one-dimensional and therefore $E_{\infty}^{0,n}$ has to be zero. But

$$0 = E_{\infty}^{0,n} \simeq E_3^{0,n} = \text{Ker } d.$$

and d is injective. But this implies that $H_I^n(R) = 0$, as required.

Now let $d = \dim R/I > 1$. By the inductive hypothesis and because of $c = \text{height } I_{R_{\mathfrak{p}}}$ for all $\mathfrak{p} \in V(I)$ (cf. Corollary 2.9) it follows that $\text{Supp}_R H_I^i(R) \subset \{\mathfrak{m}\}$ for all $i \neq c$. Therefore $H_{\mathfrak{m}}^i(C_R(I)) \simeq H_I^i(R)$ for all $c \leq i \leq n$ and zero elsewhere (cf. Proposition 1.6). By virtue of Lemma 2.2 the assumption implies that $H_I^i(R) = 0$ for all $i \neq n - 1, n$ and $i \neq c$. So it remains to show that $H_I^i(R) = 0$ for $i = n - 1, n$. Without loss of generality we may assume that R is complete since $R \rightarrow \hat{R}$ is a faithful flat extension and commutes with local cohomology.

As above let $R \xrightarrow{\sim} E$ denote a minimal injective resolution of R . Then define the complex $E_1 = \Gamma_I(E)[-c]$. By the previous observation it is up to cohomological degrees $d, d - 1$ an injective resolution of $H_I^c(R)$. Moreover let E_2 denote a minimal injective resolution of $H_I^c(R)$. The assumption on the Bass numbers in (ii) provides that

$$E_2^i = \bigoplus_{\mathfrak{p} \in V(I), h(\mathfrak{p})=i} E_R(R/\mathfrak{p}),$$

where $E_R(R/\mathfrak{p})$ denotes the injective hull of R/\mathfrak{p} . Therefore there is a comparison map of complexes $\phi : E_2 \rightarrow E_1$ such that ϕ^i is the identity for all $i \neq d$. Moreover $E_2^d = E_1^d = E(k)$, the injective hull of the residue field. By Matlis duality it follows that the endomorphism ϕ^d is given by the multiplication with a certain element $x \in R$. It is easily seen that ϕ induces in homological degree $d - 1$ and d resp. the following isomorphisms

$$H_I^{n-1}(R) \simeq 0 :_E x \text{ and } H_I^n(R) \simeq E/xE.$$

Still we have to show that both vanish. To this end let $D(\cdot) = \text{Hom}_R(\cdot, E)$ denote the Matlis functor. Then

$$D(H_I^{n-1}(R)) \simeq R/xR \text{ and } D(H_I^n(R)) \simeq 0 :_R x.$$

The associated prime ideals of R are all of dimension n . We split them into two disjoint subsets

$$U = \{\mathfrak{p} \in \text{Ass } R \mid \dim R/(I + \mathfrak{p}) = 0\} \text{ and } V = \{\mathfrak{p} \in \text{Ass } R \mid \dim R/(I + \mathfrak{p}) > 0\}.$$

First of all V can not be empty. Since otherwise $U = \text{Ass } R$ and I is an \mathfrak{m} -primary ideal in contradiction to $d = \dim R/I > 1$. Second we claim that U can not be empty. Otherwise it follows by the Hartshorne-Lichtenbaum Theorem (cf. Proposition 1.7) that $H_I^n(R) = 0$. Therefore the multiplication by x on R is injective and

$$\text{Ass}_R D(H_I^{n-1}(R)) = \text{Ass } R/xR.$$

In case xR is a proper ideal it implies that \mathfrak{p} is minimal and of height one for any $\mathfrak{p} \in \text{Ass } R/xR$. Therefore

$$0 \neq \text{Hom}_R(R/\mathfrak{p}, D(H_I^{n-1}(R))) \simeq D(H_I^{n-1}(R/\mathfrak{p}))$$

for all $\mathfrak{p} \in \text{Ass}_R R/xR$. Whence $\dim R/(\mathfrak{p} + I) = 0$ for all $\mathfrak{p} \in \text{Ass}_R R/xR$ by the Hartshorne-Lichtenbaum theorem. Because of $\text{Rad}(I + xR) = \mathfrak{m}$ this implies $d = \dim R/I \leq 1$, a contradiction.

Now define \mathfrak{a} and \mathfrak{b} resp. the intersections of all the primary components of the zero ideal of R where the corresponding primes belong to U and V resp. Because both, U and V are non-empty, \mathfrak{a} as well as \mathfrak{b} is a proper ideal in R and $0 = \mathfrak{a} \cap \mathfrak{b}$. The natural short exact sequence

$$0 \rightarrow R \rightarrow R/\mathfrak{a} \oplus R/\mathfrak{b} \rightarrow R/(\mathfrak{a} + \mathfrak{b}) \rightarrow 0$$

induces an exact sequence

$$\begin{aligned} D(H_I^{n-1}(R/(\mathfrak{a} + \mathfrak{b}))) &\rightarrow D(H_I^{n-1}(R/\mathfrak{a}))) \oplus D(H_I^{n-1}(R/\mathfrak{b}))) \rightarrow \\ &D(H_I^{n-1}(R)) \rightarrow D(H_I^{n-2}(R/(\mathfrak{a} + \mathfrak{b}))). \end{aligned}$$

In the following we want to compare the maximal members of the supports of the modules in this exact sequence. Because of $\text{Rad}(\mathfrak{a} + I) = \mathfrak{m}$ as follows by the definition of \mathfrak{a} there are the following isomorphisms

$$\begin{aligned} D(H_I^i(R/(\mathfrak{a} + \mathfrak{b}))) &\simeq D(H_{\mathfrak{m}}^i(R/(\mathfrak{a} + \mathfrak{b}))), \\ D(H_I^i(R/\mathfrak{a}))) &\simeq D(H_{\mathfrak{m}}^i(R/\mathfrak{a}))) \end{aligned}$$

for all $i \in \mathbb{Z}$. For the finitely generated R -modules on the right hand side, the so-called modules of deficiency, there are estimates of their dimension. In particular

$$\dim D(H_{\mathfrak{m}}^i(R/(\mathfrak{a} + \mathfrak{b}))) \leq n - 1, i = n - 1, n - 2, \text{ and } \dim D(H_{\mathfrak{m}}^{n-1}(R/\mathfrak{a}))) \leq n - 1$$

because $\text{height}(\mathfrak{a} + \mathfrak{b}) \geq 1$ (and every $R/(\mathfrak{a} + \mathfrak{b})$ -module has dimension at most $n - 1$). These estimates in accordance with the above short exact sequence provide the following equality about associated prime ideals

$$\text{Ass } R \cap \text{Ass } D(H_I^{n-1}(R)) = \text{Ass } R \cap \text{Ass } D(H_I^{n-1}(R/\mathfrak{b})).$$

In case of $H_I^n(R) \neq 0$ there is a prime $\mathfrak{q} \in \text{Ass } D(H_I^n(R)) = \text{Ass}(0 :_R x)$. Therefore $\dim R/\mathfrak{q} = n$ and $x \in \mathfrak{q}$. This implies that $\mathfrak{q} \in \text{Ass } R/xR = \text{Ass } D(H_I^{n-1}(R))$ and $\mathfrak{q} \in \text{Ass } D(H_I^{n-1}(R/\mathfrak{b}))$ by the previous equality. In particular $\mathfrak{b} \subset \mathfrak{q}$ since $D(H_I^{n-1}(R/\mathfrak{b}))$ is annihilated by \mathfrak{b} . Therefore $\mathfrak{q} \in V$ but this is in contradiction to $\mathfrak{p} \in \text{Ass } D(H_I^n(R))$, which means $\mathfrak{q} \in U$ by the Hartshorne-Lichtenbaum vanishing theorem. We can solve

this controversy only in case x is a unit. But this proves the vanishing of both of the local cohomology modules $H_I^i(R)$, $i = n, n - 1$, as required. \square

As an application we are able to prove Theorem 0.1 of the Introduction.

Proof. **Theorem 0.1:** Because I is a complete intersection in $V(I) \setminus \{\mathfrak{m}\}$ it follows that $IR_{\mathfrak{m}}$ is generated by c elements in $R_{\mathfrak{p}}$ for all $V(I) \setminus \{\mathfrak{m}\}$. That means the conditions (ii), (ii), and (iv) hold for any localization with respect to $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$. Therefore the equivalence of the conditions (i), (ii), and (iii) follows by virtue of Theorem 3.1. While the equivalence of (i) and (iv) is a particular case of Theorem 3.2.

Moreover, if I satisfies one of the equivalent conditions the conclusion about $H_I^i(R)$ are shown in Corollary 2.8. \square

4. EXAMPLES AND REMARKS

Let us discuss the necessity of the local conditions in Theorem 3.1. By the results of Bass (cf. [1]) a local ring is a Gorenstein ring if and only if $\dim_k \text{Ext}_R^i(k, R) = \delta_{n,i}$, $n = \dim R$. Moreover, let (R, \mathfrak{m}) be a Gorenstein ring. Then $R_{\mathfrak{p}}$, $\mathfrak{p} \in \text{Spec } R$, is also a Gorenstein ring, i.e. the Gorenstein property localizes. The following example shows that the property

$$\dim_k \text{Ext}_R^i(k, H_I^c(R)) = \delta_{d,i}, \quad d = \dim R/I,$$

does not localize to the corresponding statement for $H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})$, $\mathfrak{p} \in V(I)$.

Example 4.1. Let k be an arbitrary field. Let $R = k[[x_0, x_1, x_2, x_3, x_4]]$ denote the formal power series ring in five variables over k . Let $I = (x_0, x_1) \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4)$. Then $c = \text{height } I = 2$ and $\dim_k \text{Ext}_R^i(k, H_I^2(R)) = \delta_{3,i}$, $i \in \mathbb{N}$. Moreover $H_I^i(R) \neq 0$ for all $i \neq 2, 3$.

Proof. Obviously we have $c = 2$. Put

$$I_1 = (x_0, x_1) \cap (x_1, x_2), \quad I_2 = (x_2, x_3) \cap (x_3, x_4), \quad J = I_1 + I_2.$$

By the aid of the Mayer-Vietoris sequence with respect to I_1 and I_2 it follows that

$$H_I^i(R) = 0 \text{ for } i \neq 2, 3 \text{ and } H_I^3(R) \simeq H_J^4(R).$$

Moreover $J = J_1 \cap J_2$ with $J_1 = (x_0, x_1, x_3, x_4)$ and $J_2 = (x_1, x_2, x_3)$, as it is easily seen. A second use of the Mayer-Vietoris sequence with respect to J_1 and J_2 gives a short exact sequence

$$0 \rightarrow H_{J_1}^4(R) \rightarrow H_J^4(R) \rightarrow E \rightarrow 0.$$

Recall that $J_1 + J_2 = \mathfrak{m}$ and $H_{\mathfrak{m}}^5(R) \simeq E$, where \mathfrak{m} denotes the maximal ideal of R and $E = E_R(R/\mathfrak{m})$. Because of $H_{J_1}^i(R) = 0$ for all $i \neq 4$ the truncation process provides a short exact sequence

$$0 \rightarrow H_{J_1}^4(R) \rightarrow E_R(R/J_1) \rightarrow E \rightarrow 0.$$

Localizing both exact sequences at x_2 implies the following isomorphisms

$$H_J^4(R)_{x_2} \simeq H_{J_1}^4(R)_{x_2} \simeq E_R(R/J_1).$$

Recall that x_2 acts bijectively on $E_R(R/J_1)$. Moreover there is the naturally defined exact sequence

$$0 \rightarrow H_{x_2}^0(H_J^4(R)) \rightarrow H_J^4(R) \xrightarrow{f} H_J^4(R)_{x_2} \rightarrow H_{x_2}^1(H_J^4(R)) \rightarrow 0.$$

In the next step we show that f is an isomorphism. Therefore we have to show that $H_{x_2}^i(H_J^4(R)) = 0$ for $i = 0, 1$. To this end consider the short exact sequence

$$0 \rightarrow H_{x_2}^1(H_J^{i-1}(R)) \rightarrow H_{x_2R+J}^i(R) \rightarrow H_{x_2}^0(H_J^i(R)) \rightarrow 0$$

(cf. [12, Corollary 1.4]). Because of the equality $\text{Rad}(x_2R + J) = (x_1, x_2, x_3)R$ it follows that $H_{x_2R+J}^i(R) = 0$ for all $i \neq 3$. With this in mind the previous short exact sequence implies that f is an isomorphism. This means that $H_I^3(R) \simeq H_J^4(R) \simeq E_R(R/J_1)$ and therefore $H_I^3(R)$ is an injective R -module.

Finally consider the spectral sequence

$$E_2^{p,q} = \text{Ext}_R^p(k, H_I^q(R)) \implies E_\infty^{p+q} = \text{Ext}_R^{p+q}(k, R).$$

Because of $\text{Ext}_R^p(k, H_I^3(R)) = 0$ for all $p \in \mathbb{Z}$ it degenerates to isomorphisms

$$\text{Ext}_R^p(k, H_I^2(R)) \simeq \text{Ext}_R^{p+2}(k, R) \text{ for all } p \in \mathbb{Z}.$$

Because R is a Gorenstein ring it follows that $\dim_k \text{Ext}_R^p(k, H_I^2(R)) = \delta_{3,p}$ as required. \square

Another problem related to our considerations is the finiteness of the Bass numbers of $H_I^c(R)$. Recall that the Bass numbers of a finitely generated R -module are always finite (cf. [1]). This is not the case for the Bass numbers of $H_I^c(R)$, $c = \text{height } I$.

Example 4.2. Let k denote a field and $R = k[[x, y, u, v]]/(xu - yv)$, where $k[[x, y, u, v]]$ denotes the power series ring in four variables over k . Let $I = (u, v)R$. Then $\dim R = 3$, $\dim R/I = 2$ and $c = 1$. It follows that $H_I^i(R) = 0$ for $i \neq 1, 2$. The truncation complex with the short exact sequence (cf. 1.1)

$$0 \rightarrow H_I^c(R)[-c] \rightarrow \Gamma_I(E) \rightarrow C_R(I) \rightarrow 0$$

induces an injection

$$0 \rightarrow \text{Hom}_R(k, H_I^2(R)) \rightarrow \text{Ext}_R^2(k, H_I^1(R)).$$

Hartshorne (cf. [4, §3]) has shown that the socle of $H_I^2(R)$ is not a finite dimensional k -vector space. Therefore, the second Bass number of $H_I^1(R)$ is infinite.

As mentioned at the beginning a set-theoretic complete intersection is a cohomologically complete intersection. The converse is not true. Let $\mathfrak{p} \subset k[x_0, x_1, x_2, x_3]$ a homogeneous prime ideal of dimension two. Then \mathfrak{p} in $R = k[x_0, x_1, x_2, x_3]_{(x_0, x_1, x_2, x_3)}$ is always a cohomologically complete intersection because $H_{\mathfrak{p}}^3(R) = 0$ by [5, Theorem 7.5]. In the following we will remark that the property of being a set-theoretic complete intersection is – by virtue of Theorems 3.1 and 3.2 – also completely encoded in the local cohomology $H_I^c(R)$, $c = \text{height } I$.

The following result is a particular case of [7, section 0] or, with more details, in [6, 1.1.4]. For the sake of completeness we include a proof.

Lemma 4.3. *Let (R, \mathfrak{m}) denote a local ring. Let $I \subset R$ denote an ideal. Let $f_1, \dots, f_c, c = \text{height } I$, be a regular sequence contained in I . Then the following conditions are equivalent:*

- (i) $\text{Rad } I = \text{Rad}(f_1, \dots, f_c)R$.
- (ii) I is a cohomologically complete intersection and f_1, \dots, f_c is a regular sequence on $\text{Hom}_R(H_I^c(R), E)$.

Proof. We show (ii) \implies (i). It is easy to see that $\text{cd } I = 0$ if and only if $\text{ara } I = 0$. Now let $f \in I$ denote an R -regular element. Then the multiplication map by f induces an exact sequence

$$0 \rightarrow H_I^{c-1}(R/fR) \rightarrow H_I^c(R) \xrightarrow{f} H_I^c(R) \rightarrow H_I^c(R/fR) \rightarrow 0$$

and the vanishing $H_I^i(R/fR) = 0$ for all $i \neq c-1, c$. Then $\text{cd } I(R/fR) = c-1$ if and only if f is regular on $\text{Hom}_R(H_I^c(R), E)$. So an induction on c proves the claim.

The converse (i) \implies (ii) follows by a similar consideration. \square

Because of the previous arguments it would be of some interest to understand the structure of $H_I^c(R)$, $c = \text{height } I$, in a better way.

Example 4.4. Let $R = k[x_0, x_1, x_2, x_3]_{(x_0, x_1, x_2, x_3)}$ and let \mathfrak{p} be the defining ideal of the rational quartic given parametrically by (s^4, s^3t, st^3, t^4) in \mathbb{P}_k^3 . Therefore

$$\mathfrak{p} = (x_0x_3 - x_1x_2, x_1^3 - x_0^2x_2, x_1^2x_3 - x_0x_2^2, x_1x_3^2 - x_2^3).$$

By the above remark $\text{cd } \mathfrak{p} = 2$, so that \mathfrak{p} is cohomologically a complete intersection. Let k a field of positive characteristic p . Let $n, r, s \in \mathbb{N}$ such that $p^n = 3r + 4s$. Then it was shown (cf. [9, II.2.(ii)]) that $\mathfrak{p} = \text{Rad}(F, G)$, where

$$F = x_1^3 - x_0^2x_2, G = (x_2^4 - x_0x_3^3)^{p^n} + 3x_0^rx_1^sx_2^{4r+2s}x_3^{p^n}(x_0^rx_1^sx_3^{p^n} - x_2^{4r+2s}).$$

Therefore F, G is a regular sequence on $\text{Hom}_R(H_{\mathfrak{p}}^2(R), E)$. It is an open problem whether there is such a regular sequence in the case of $\text{char } k = 0$.

Related to the characterization of a Gorenstein ring and the results in Theorem 3.2 there is the following problem concerning the Bass numbers.

Problem 4.5. A Gorenstein ring (R, \mathfrak{m}) is a Cohen-Macaulay ring of type 1. That means the following statement about the Bass numbers. Suppose that $\dim_k \text{Ext}_R^i(k, R) = \delta_{d,i}$ for all $i \leq d = \dim R$. Then $\dim_k \text{Ext}_R^i(k, R) = \delta_{d,i}$ for all $i \in \mathbb{Z}$. We do not know whether it will be sufficient to replace the condition (ii) in Theorem 3.2 by

$$\dim_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) = \delta_{h(\mathfrak{p}),i} \text{ for all } i \leq h(\mathfrak{p}).$$

REFERENCES

- [1] H. BASS: *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1983), 18-29.
- [2] A. GROTHENDIECK: ‘Local cohomology’, Notes by R. Hartshorne, Lect. Notes in Math., **20**, Springer, 1966.
- [3] R. HARTSHORNE: ‘Residues and Duality’, Lect. Notes in Math., **41**, Springer, 1967.
- [4] R. HARTSHORNE: *Affine duality and cofiniteness*, Inventiones Math. **9** (1970), 145-164.
- [5] R. HARTSHORNE: *Cohomological dimension of algebraic varieties*, Ann. Math. **88** (1968), 403-405.
- [6] M. HELLUS: *Local Cohomology and Matlis Duality*, Habilitationsschrift, Leipzig, 2006, available from <http://www.math.uni-leipzig.de/~hellus/HabilitationsschriftOhneDeckblatt.pdf>
- [7] M. HELLUS: *On the associated primes of Matlis duals of top local cohomology modules*, Communications in Algebra **33** (2005), 3997-4009.
- [8] M. HELLUS, J. STÜCKRAD: *On endomorphism rings of local cohomology modules*, Proc. Amer. Math. Soc. **136** (2008), 2333-2341.
- [9] H. ROLOFF, J. STÜCKRAD: *Bemerkungen über Zusammenhangseigenschaften und mengentheoretische Darstellung projektiver Varietäten*, Beitr. Algebra Geom. **8** (1979), 125-131.

- [10] P. SCHENZEL: *Proregular sequences, Local Cohomology, and Completion*, Math. Scand. **92** (2003), 181-180.
- [11] P. SCHENZEL: *On birational Macaulayfications and Cohen-Macaulay canonical modules*, J. Algebra **275** (2004), 751-770.
- [12] P. SCHENZEL: *On the use of local cohomology in algebra and geometry*. In: Six Lectures in Commutative Algebra, Progress in Math. Vol. 166, Birkhäuser, 1998, pp. 241-292.
- [13] C. WEIBEL: 'An Introduction to Homological Algebra', Cambr. Univ. Press, 1994.

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